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EXTREMAL CONTROL IN A NONLINEAR DIFFERENTIAL GAME

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N. N. KRASOVSKII
(Sverdlovsk)

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We consider the game problem of the encounter of a conflict-controlled phase point with a given set. We prove sufficient conditions for the successful completion of a nonlinear game of encounter. These conditions are based on the idea of minimax extremal aiming [1]. The given aiming is realized here on the basis of absorption sets [2]. These sets are constructed with the aid of auxiliary motions generated by program controls which are represented by suitable Borel measures in accordance with the well known techniques [3] of generalized solutions of ordinary differential equations.

1. Statement of the problem. We consider a controlled system described by the vector differential equation

$$\dot{x} = f(t, x, u, v) \tag{1.1}$$

Here x is the system's n-dimensional phase vector; u and v are r-dimensional vector controls of the first and second players, respectively, constrained by the conditions $u \in P$, $v \in Q$, where P and Q are bounded closed sets. The function $f(t, x, u, v)$ is assumed continuous for all argument values to be considered and satisfies a Lipschitz condition in x in every bounded region of the space $\{x\}$. Furthermore, the following conditions for the continuability of the solutions $x[t]$ for Eq. (1.1) are assumed to be fulfilled. Let $F(t, x) = \text{co}^* \{f(t, x, u, v): u \in P, v \in Q\}$, where $\text{co}^* \{f\}$

denotes the closed convex hull of a certain set $\{f\}$ of vectors f. Then we take it that for given $t = t_0$ and for a bounded region G in the space $\{x\}$, for any $\vartheta > t_0$ we can find a number $\beta(t_0, G, \vartheta)$ such that every solution $x(t)$ of the contingent equations [4]

$$\dot{x}(t) \in F(t, x(t)) \tag{1.2}$$

under the conditions $x(t_*) \in G$, $(t_0 \leqslant t_* \leqslant \vartheta)$, satisfies the inequality

$$\| x(t) \| \leqslant \beta(t_0, G, \vartheta) \tag{1.3}$$

for all $t_* \leqslant t \leqslant \vartheta$. Here and below $\| x \|$ denotes the Euclidean norm of vector x. By the problem's hypotheses, in the space $\{x\}$ we are given a closed set M which constitutes the first player's target.

We define strategies $U \div u(t, x)$ by means of functions $u(t, x)$ which associate a certain vector u with every possible position $\{t, x\}$ (*). Any strategy $U \div u(t, x)$ constrained only by the condition $u(t, x) \in P$ for all values of arguments t and x being considered, is assumed to be admissible. Let $[\tau_i, \tau_{i+1})$ $(i = 0, 1, 2, \ldots)$ be a certain system Δ of semi-intervals covering the semiaxis $[t_*, \infty)$ $(\tau_0 = t_*)$. Let us choose some measurable realization $v[t]$ $(t_* \leqslant t < \infty)$ of the second player's control v, constrained by the condition $v[t] \in Q$. An absolutely continuous function $x_\Delta[t]$, satisfying the initial condition $x_\Delta[t_*] = x_*$ and the equation

$$x_\Delta^{\cdot}[t] = f(t, x_\Delta[t], u(\tau_i, x_\Delta[\tau_i]), v[\cdot]) \tag{1.4}$$

for almost all $\tau_i \leqslant t < \tau_{i+1}$ $(i = 0, 1, 2, \ldots)$ is called Euler's polygonal line $x_\Delta[t] = x_\Delta[t, t_*, x_*, U, v[\cdot]]$. Here $u(t, x)$ is precisely that function which corresponds to the strategy U fixed in the notation $x_\Delta[t, t_*, x_*, U, v[\cdot]]$. By a motion $x[t] = x[t, t_*, x_*, U]$ of system (1.1) under a strategy $U \div u(t, x)$ chosen by the first player we mean every function $x[t]$ $(t \geqslant t_0)$ which on any finite interval $[t_0, \vartheta]$ is the uniform limit of a certain suitable sequence of Euler's polygonal lines $x_{\Delta(k)}[t] = x_{\Delta(k)}[t, t_*, x_*^{(k)}, U, v^{(k)}[\cdot]]$ $(k = 1, 2, \ldots)$ under the condition that $\lim [\sup_i (\tau_{i+1}^{(k)} - \tau_i^{(k)})] = 0$ as $k \to \infty$. In these terms the first player's problem of leading the phase point $x[t]$ onto set M is stated in the following manner.

Problem 1.1. For a given initial position $\{t_0, x_0\}$ find a strategy $U^\circ \div u^\circ(t, x)$ which would guarantee the encounter of any motion $x[t] = x[t, t_0, x_0, U^\circ]$ with set M, i.e. would ensure for any such motion $x[t] = x[t, t_0, x_0, U^\circ]$ the fulfillment of the condition

$$x[t^*] \in M \tag{1.5}$$

at some instant $t = t^* < \infty$, possibly, at its own instant t^* $(x[\cdot])$ for each motion $x[t]$ $(t \geqslant t_0)$.

2. Program controls. Let us define the class of auxiliary controls η to be considered and of the auxiliary motions $x(t)$ generated by them. On the semi-interval $T = [t_0, \vartheta)$ we identify the program controls $\eta = \eta(dt, du, dv)$ with regular Borel measures $\eta(dt, du, dv)$ defined in the space $\{t\} \times \{u\} \times \{v\}$, concentrated on the set $T \times P \times Q$, and normed in such a way that the equality

$$\eta(T^* \times P \times Q) = t^* - t_* \tag{2.1}$$

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*) Editor's Note. The symbol $\div$ denotes the correspondence between the strategy and the function prescribing this strategy.

is fulfilled for any semi-interval $[t_*, t^*) = T^* \subset T$, i.e. the measure of the set $T^* \times P \times Q$ equals the length of the semi-interval T^*. The measures $\eta\ (dt,\ du,\ dv)$ will be treated as elements of the space of linear functionals $\varkappa_\eta\ [\varphi]$ defined on continuous functions $\varphi\ (t,\ u,\ v)$ so that in accordance with [5]

$$\varkappa_\eta [\varphi] = \iiint \varphi(t, u, v)\, \eta(dt, du, dv) \tag{2.2}$$

Everywhere in what follows the weak topology in the space of measures $\eta\ (dt,\ du,\ dv)$ is to be understood in the sense of the weak topology in the space of the functionals $\varkappa_\eta\ [\varphi]$ in (2.2) with respect to the original space $\{\varphi\}$ of continuous functions $\varphi\ (t,\ u,\ v)$. The program motion $x(t) = x(t,\ t_*,\ x_*,\ \eta)$ $(t_* \leqslant t \leqslant \vartheta)$ generated by the program control $\eta = \eta\ (dt,\ du,\ dv)$ from the position $\{t_*, x_*\}$, is defined as the solution of the following integral equation:

$$x(t) = x_* + \int_{t_*}^{t} \int_P \int_Q f(\tau, x(\tau), u, v)\, \eta(d\tau, du, dv) \tag{2.3}$$

By standard methods of the theory of ordinary differential equations we can establish that under the assumptions made on the right-hand side $f\ (t,\ x,\ u,\ v)$ of Eq.(1.1) the integral Eq. (2.3), for every choice of position $\{t_*,\ x_*\}$ and of the admissible control $\eta = \eta\ (dt,\ du,\ dv)$, has a unique absolutely-continuous solution $x\ (t)$ continuable for all values of $t \in [t_*,\ \vartheta\ ]$. For any choice of a bounded region G in space $\{x\}$ all program motions $x\ (t) = x\ (t,\ t_*,\ x_*,\ \eta)(t_* \leqslant t \leqslant \vartheta)$ turn out to be uniformly bounded and equicontinuous for all possible $t_* \in [t_0,\ \vartheta)$, $x_* \in G$ and $\eta = \eta\ (dt,\ du,\ dv)$. If some sequence of controls $\{\eta^{(k)}\}$ $(k = 1, 2, \ldots)$ converges weakly as $k \to \infty$ to a control η^*, then the corresponding sequence of motions $x^{(k)}\ (t) = x\ (t,\ t_*,\ x_*,\ \eta^{(k)})$ converges uniformly to the motion $x^*\ (t) = x\ (t,\ t_*,\ x_*,\ \eta^*)$. Finally, the motions $x\ (t) = x\ (t,\ t_*,\ x_*,\ \eta)$ are uniformly continuous with respect to the initial conditions for $\{t_*,\ x_*\}$ from any bounded region $t_0 \leqslant t_* \leqslant \vartheta$, $x_* \in G$, and, moreover, are equicontinuous in t and η.

Regular Borel measures $\mu\ (dt,\ du)$ defined in the space $\{t\} \times \{u\}$, concentrated on the set $T \times P$, and normed so that the equality

$$\mu(T^* \times P) = t^* - t_* \tag{2.4}$$

is fulfilled for any semi-interval $[t_*,\ t^*) = T^* \subset T$, i.e. the measure of the set $T^* \times P$ equals the length of the semi-interval T^*, are called the program controls $\mu = \mu\ (dt,\ du)$ of the first player on the semi-interval $[t_0,\ \vartheta\ )$. In what follows we assume that a certain set $\{\mu\ (dt,\ du)\}$ of controls $\mu = \mu\ (dt,\ du)$ is fixed. These controls $\mu = \mu\ (dt,\ du) \in \{\mu\ (dt,\ du)\}$ are called the admissible program controls of the first player on the semi-interval $[t_0,\ \vartheta\ )$. In particular, the set of all regular Borel measures $\mu\ (dt,\ du)$, satisfying condition (2.4), can be chosen as the set $\{\mu\ (dt,\ du)\}$ of the first player's admissible program controls $\mu = \mu\ (dt,\ du)$.

In what follows the program controls η, μ, and the program motions $x\ (t)$ generated by them, will often be considered on time subsets from the semi-interval $[t_0,\ \vartheta\ )$ or from the interval $[t_0,\ \vartheta\ ]$, respectively. In order to emphasize this fact, we shall sometimes write the notation of the time subset selected after the notation of the corresponding control or motion. For example, the entry η, $(t_* \leqslant t < t^*)$ or η, $[t_*,\ t^*)$ means that the measure $\eta\ (dt,\ du,\ dv)$ forming a given program control, is to be considered on a subset of $T^* \times P \times Q$, where $T^* = [t_*, t^*)$. The part of the control η, $(t_0 \leqslant t < \vartheta)$, specified by the measure $\eta\,(dt, du, dv)$ on the whole set $T \times P \times Q$, being considered as the control $\eta = \eta\,(dt, du, dv)$ $(t_* \leqslant t < t^*)$ on the semi-interval $T^* = [t_*, t^*) \subset [t_0, \vartheta) = T$, will be called the control segment η, $[t_0, \vartheta)$ corresponding to this semi-interval $[t_*, t^*)$.

Further, we assume as fixed a certain set $\{\eta\,(dt, du, dv)\}$ of admissible program controls $\eta\,(dt, du, dv)$. We assume that this set is convex and weakly-closed. Since the set of all possible regular Borel measures $\eta\,(dt, du, dv)$, satisfying condition (2.1), is weakly compact in itself, the weakly-closed set $\{\eta\,(dt, du, dv)\}$ of all admissible program controls $\eta\,(dt, du, dv)$ also is weakly compact in itself.

Every weakly-closed collection $\{\eta\,(dt, du, dv), [t_*, t^*)\}_{\Pi}$ is called the second player's program on some semi-interval $[t_*, t^*) = T^*$, comprised of admissible program controls $\eta = \eta\,(dt, du, dv)$ considered on the set $T^* \times P \times Q$ and satisfying the following condition: whatever be the first player's admissible program control $\mu = \mu\,(dt, du) \in \{\mu\,(dt, du)\}$, in the program $\{\eta\,(dt, du, dv), [t_*, t^*)\}_{\Pi}$ we can find at least one control $\eta\,(dt, du, dv)$ matched with the measure $\mu\,(dt, du)$ by the condition

$$\eta\,(A \times B \times Q) = \mu\,(A \times B) \tag{2.5}$$

which must be fulfilled whatever be the measurable sets $A \subset T^*$ and $B \subset P$.

Let us fix a certain value $x = x_*$ and a certain n-dimensional vector s. The program $\{\eta\,(dt, du, dv), [t_*, t^*), x_*, s\}_{\Pi}$ is said to be extremal to $\{x_*, s\}$ on the semi-interval $[t_*, t^*)$ if it forms a convex and, as is every program of the second player, weakly-closed set of program controls $\eta\,(dt, du, dv)$ and satisfies the condition

$$\int_{t_*}^{t^*} [\min_{u \in P} \max_{v \in Q} s' f\,(t, x_*, u, v)]\, dt \leqslant \tag{2.6}$$

$$\min_{\eta \in \{\eta\}_{\Pi}} \Big[\int_{t_*}^{t^*} \int_P \int_Q s' f\,(t, x_*, u, v)\, \eta\,(dt, du, dv) \Big] + \|s\|\, o_G\,(t^* - t_*)$$

where $\{\eta\}_{\Pi} = \{\eta, [t_*, t^*), x_*, s\}_{\Pi}$. Here and below the prime denotes transposition and, so, the symbol $s'f$ denotes the scalar product of vectors s and f. The symbol $o_G\,(\delta)$ denotes a small quantity of order higher than δ, moreover, the estimate $o_G\,(\delta)$ is assumed to be uniform in $t_* \in [t_0, \vartheta]$ and $\{x_*\} \in G$, where G is any preselected bounded region in the space $\{x\}$. The set $\{\eta\,(dt, du, dv)\}$ of admissible program controls $\eta\,(dt, du, dv)$ is assumed to be so complete that at least one extremal program $\{\eta\,(dt, du, dv), [t_*, t^*), x_*, s\}_{\Pi}$ can be set up for any choice of $[t_*, t^*) \subset [t_0, \vartheta)$, x_* and s. Furthermore, we assume that the control $\eta\,(dt, du, dv)\, [t_*, \vartheta)$, made up from segments of the two admissible controls $\eta\,(dt, du, dv)$, $[t_*, t^*)$, $\eta\,(dt, du, dv)$, $[t^*, \vartheta)$, is once again an admissible control. The completeness condition stated is fulfilled in every case if as the set of admissible program controls we choose all possible regular Borel measures satisfying condition (2.1), because then for any choice of x_* and s and of the semi-interval $[t_*, t^*)$ there exists at least one extremal program $\{\eta\,(dt, du, dv), [t_*, t^*), x_*, s\}_{\Pi}$. This program can be obtained, for example, in the following manner. We choose some admissible control $\mu = \mu\,(dt, du)$ and we consider the part of it corresponding to the semi-interval $[t_*, t^*)$. We can construct a Borel measure $\eta\,(dt, du, dv)$ matched with the chosen control $\mu\,(dt, du)$ by condition (2.5) and simultaneously satisfying the condition

$$\int_{t_*}^{t^*}\int_P\int_Q s'f(t, x_*, u, v)\,\eta(dt, du, dv) = \int_{*}^{*}\int_P [\max_{v\in Q} s'f(t, x_*, u, v)]\,\mu(dt, du) \quad (2.7)$$

If now with each admissible control $\mu = \mu(dt, du)$ $(t_* \leqslant t < t^*)$ we compare all possible regular Borel measures $\eta(dt, du, dv)$ $(t_* \leqslant t < t^*)$ matched with it by condition (2.5), satisfying condition (2.7), and we take the weakly-closed convex hull of all such measures $\eta(dt, du, dv)$ corresponding to all possible admissible controls $\mu = \mu(dt, du) \in \{\mu(dt, du)\}$, then we obtain the needed extremal program $\{\eta(dt, du, dv), [t_*, t^*), x_*, s\}_\Pi$ satisfying condition (2.6).

The formal definitions presented can be meaningfully clarified as follows. Suppose that some position $\{t_*, x_*\}$ has been realized. Then the second player is allowed to freeze this position mentally and to choose mentally a certain program $\{\eta(dt, du, dv), [t_*, \vartheta)\}_\Pi$ of his own for the future $t_* \leqslant t < \vartheta$. The second player informs the first player of his choice. After this the first player can choose mentally an arbitrary admissible program control $\eta = \eta(dt, du, dv)$ $(t_* \leqslant t \leqslant \vartheta)$ contained in this program $\{\eta(dt, du, dv), [t_*, \vartheta)\}$ which the second player has planned. The control $\eta = \eta(dt, du, dv)$ chosen in this manner determines a certain imaginary program motion $x(t) = x(t, t_*, x_*, \eta)$ by (2.3). Here, in each case, the first player's choice proves to be so broad that he has the possibility of at least choosing from the program $\{\eta(dt, du, dv), [t_*, \vartheta)\}_\Pi$ an admissible control $\eta = \eta(dt, du, dv)$ matched by condition (2.5) with any admissible program control $\mu = \mu(dt, du) \in \{\mu(dt, du)\}$ at which the first player wishes to fix his own choice. Meanwhile, the second player also has a sufficiently large selection of programs $\{\eta(dt, du, dv), [t_*, \vartheta)\}_\Pi$, because he has the possibility of sticking at any program $\{\eta(dt, du, dv), [t_*, \vartheta)\}_\Pi$ constructed on the basis of one or of some set of extremal programs $\{\eta(dt, du, dv), [\tau_*, \tau^*), x_*, s\}_\Pi$ whose choice, under the condition posed, is sufficiently wide. It is clear that in the proposed construction these meaningful concepts have been coded in the form of measure-controls mixing the ordinary controls u and v. Thus, the proposed programs $\{\eta(dt, du, dv), [t_*, \vartheta)\}_\Pi$ and controls $\eta = \eta(dt, du, dv) \in \{\eta, [t_*, \vartheta)\}_\Pi$, related to the controls $\mu = \mu(dt, du)$ by condition (2.5), indeed do bear the character of the formalizations corresponding to the meaningful concepts of a program control and of the corresponding concepts of minimax program absorption from [1].

It should be stressed that the procedure described for selecting the program $\{\eta(dt, du, dv), [t_*, \vartheta)\}_\Pi$ and the control $\eta(dt, du, dv) \in \{\eta(dt, du, dv), [t_*, \vartheta)\}_\Pi$ was not provided in the conditions of the initial Problem 1.1, and this procedure is of an auxiliary nature. The presentation of the auxiliary program motions $x(t)$ allow us, however, to evolve a certain auxiliary extremal construction which serves as the foundation for forming the control u, and solving Problem 1.1 at every position $\{t_*, x_*\}$ even during the actual game. Moreover, the motions $x[t] = x[t, t_0, x_0, U]$ are now formed on the feedback principle in accordance with the scheme described in Sect. 1 and based on the construction made up from the Euler's polygonal lines (1.4).

3. Auxiliary program problems. We consider two auxiliary problems of optimal program control. On their foundation we shall formulate conditions for the regularity of the original game, sufficient for the successful solving of Problem 1.1 by the first player. The solutions of these auxiliary problems furnish us with the concept of absorption sets [2]. These sets constitute the basis of the extremal construction which determines the strategy U° solving Problem 1.1 in the regular case. By the symbol $\rho\,(x, M)$ we denote the Euclidean distance from a point x to a set M. The first auxiliary problem is the problem of an optimal program control η minimizing the quantity $\rho\,(x\,(\vartheta), M)$.

Problem 3.1. Given a position $\{t_*, x_*\}$, a number $\vartheta > t_*$, and a program $\{\eta\,(dt, du, dv), [t_*, \vartheta)\}_\Pi$, find the optimal admissible program control

$$\eta^\circ (dt, du, dv\,|\,[t_*, \vartheta), x_*, \{\eta\}_\Pi) \in \{\eta\,(dt, du, dv), [t_*, \vartheta)\}_\Pi$$

which satisfies the condition

$$\rho\,(x\,(\vartheta, t_*, x_*, \eta^\circ), M) = \min_{\eta\in\{\eta\}_\Pi}\rho\,(x\,(\vartheta, t_*, x_*, \eta), M) \tag{3.1}$$

The program motion $x\,(t) = x\,(t, t_*, x_*, \eta^\circ)$, generated by the optimal control $\eta^\circ = \eta^\circ\,(dt, du, dv\,|\,[t_*, \vartheta), x_*, \{\eta\}_\Pi)$, is called the optimal program motion solving Problem 3.1 and is denoted by the expression $x^\circ\,(t) = x^\circ\,(t, t_*, x_*, \eta^\circ\,|\vartheta, \{\eta\}_\Pi)$. From the condition that the collection of admissible controls η, making up the program $\{\eta\,(dt, du, dv), [t_*, \vartheta)\}_\Pi$, forms a set weakly compact in itself, we conclude that for each choice of position $\{t_*, x_*\}$, of number $\vartheta > t_*$ and of program $\{\eta(dt, du, dv), [t_*, \vartheta)\}_\Pi$, Problem 3.1 has a solution $\eta^\circ\,(dt, du, dv)$. Indeed, let us examine some minimizing sequence of controls $\eta^{(k)} \in \{\eta, [t_*, \vartheta)\}_\Pi$ $(k = 1, 2, \ldots)$ which satisfies the condition

$$\lim_{k\to\infty} \rho\,(x\,(\vartheta, t_*, x_*, \eta^{(k)}), M) = \inf_{\eta\in\{\eta\}_\Pi}\rho\,(x\,(\vartheta, t_*, x_*, \eta), M) \tag{3.2}$$

From this sequence we can select a weakly convergent subsequence $\eta^{(k_j)}$ $(j = 1, 2, \ldots)$, for which, as we have noted above, the corresponding sequence of motions $x^{(k_j)}\,(t) = x\,(t, t_*, x_*, \eta^{(k_j)})$ converges uniformly on the interval $[t_*, \vartheta]$ to the motion $x^*\,(t) = x\,(t, t_*, x_*, \eta^*)$ (2.3) generated by the admissible control $\eta^* = \eta^*\,(dt, du, dv) \in \{\eta\,(dt, du, dv), [t_*, \vartheta)\}_\Pi$, which is a weak limit for the sequence $\eta^{(k_j)}\,(dt, du, dv)$. But, by definition of $x^*\,(t)$, from (3.2) follows the equality

$$\rho\,(x^*\,(\vartheta), M) = \min_{\eta\in\{\eta\}_\Pi}\,\rho\,(x\,(\vartheta, t_*, x_*, \eta), M) \tag{3.3}$$

which also proves that the program motion $x^*\,(t)$ is the optimal program motion solving Problem 3.1, and the admissible program control η^* generating it, is of the optimal program control $\eta^\circ\,(dt, du, dv\,|\,[t_*, \vartheta), x_*, \{\eta\}_\Pi)$ for this problem. Thus, we have verified the existence of a solution of Probem 3.1. The second auxiliary problem is the problem of an optimal program control η which supplies the maximin to the quantity $\rho\,(x(\vartheta),$

Problem 3.2. Given a position $\{t_*, x_*\}$ and a number $\vartheta > t_*$, find the optimal admissible program control $\eta^\circ(dt, du, dv\,|\,[t_*, \vartheta), x_*)^\circ$ which satisfies the condition

$$\rho\,(x\,(\vartheta, t_*, x_*, \eta^\circ), M) = \max_{\{\eta\}_\Pi}\min_{\eta\in\{\eta\}_\Pi}\rho\,(x\,(\vartheta, t_*, x_*, \eta), M) \tag{3.4}$$

and $\eta^\circ\,(dt, du, dv)^\circ \in \{\eta\,(dt, du, dv), [t_*, \vartheta), x_*\}_\Pi^\circ$, where $\{\eta\,(dt, du, dv), [t_*, \vartheta), x_*\}_\Pi^\circ$ is that program which maximizes the right-hand side of equality (3.4).

The program motion $x\,(t) = x\,(t, t_*, x_*, \eta^\circ)$ generated by the optimal control $\eta^\circ = \eta^\circ\,(dt, du, dv\,|\,[t_*, \vartheta), x_*)^\circ$ is called the optimal program motion solving Problem 3.2 and is denoted by the expression $x^\circ\,(t)^\circ = x^\circ\,(t, t_*, x_*, \eta^\circ\,|\vartheta)^\circ$.

First of all it is necessary to verify the existence of the solution $\eta^{\circ}(dt, du, dv \mid [t_*, \vartheta), x_*)^{\circ}$ of Problem 3.2. To do this, according to what has preceded, it suffices to verify the existence of the program $\{\eta(dt, du, dv), [t_*, \vartheta), x_*\}_{\Pi}^{\circ}$ maximizing the right-hand side of (3.4). Let us show that such an admissible program $\{\eta, [t_*, \vartheta), x_*\}_{\Pi}^{\circ}$ exists for every choice of $\{t_*, x_*\}$ and $\vartheta > t_*$. Indeed, we consider any maximizing sequence of programs $\{\eta, [t_*, \vartheta)\}_{\Pi}^{(k)}$ $(k = 1, 2, \ldots)$ which satisfies the condition

$$\lim_{k\to\infty} (\min\nolimits_{\eta\in\{\eta\}_{\Pi}^{(k)}} \rho(x(\vartheta, t_*, x_*, \eta), M)) = \sup\nolimits_{\{\eta\}_{\Pi}} (\min\nolimits_{\eta\in\{\eta\}_{\Pi}} \rho(x(\vartheta, t_*, x_*, \eta), M)) \tag{3.5}$$

We consider all possible weakly-convergent subsequences $\{\eta^{(k_j)}\}$ of admissible controls $\eta^{(k_j)} \in \{\eta, [t_*, \vartheta)\}_{\Pi}^{(k_j)}$ $(j = 1, 2, \ldots)$. We consolidate the weak limits $\eta(dt, du, dv)$ of all such subsequences into a certain set $\{\eta(dt, du, dv)\}^*$. As a consequence of the weak compactness in itself of the set $\{\eta(dt, du, dv)\}$ of all admissible controls $\eta(dt, du, dv)$, each of the weak limits $\eta(dt, du, dv)$ being considered is once more an admissible program control. Thus we have obtained a certain set $\{\eta(dt, du, dv)\}^*$ of admissible program controls. It is not difficult to verify that this set satisfies the requirements made of the program on the semi-interval $[t_*, \vartheta)$. Thus, we have constructed a certain program $\{\eta(dt, du, dv), [t_*, \vartheta)\}_{\Pi}^*$.

It remains to check that this program is a maximizing one for (3.4). We assume the contrary. Then among the admissible program controls $\eta \in \{\eta, [t_*, \vartheta)\}_{\Pi}^*$ we can find a control η^* which generates a motion $x^*(t) = x(t, t_*, x_*, \eta^*)$ satisfying the condition

$$\rho(x^*(\vartheta), M) < \lim_{k\to\infty} {}_{\eta\in\{\eta\}_{\Pi}^{(k)}} (\min \rho(x(\vartheta, t_*, x_*, \eta), M)) \tag{3.6}$$

But, by the construction of the program $\{\eta\}_{\Pi}^*$, for the measures $\eta^* = \eta^*(dt, du, dv)$ we can find a subsequence, converging weakly to it, of admissible controls $\eta^{(k_j)} = \eta^{(k_j)}(dt, du, dv) \in \{\eta\}_{\Pi}^{(k_j)}$ $(j = 1, 2, \ldots)$. The sequence of program motions $x^{(k_j)}(t) = x(t, t_*, x_*, \eta^{(k_j)})$ from (2.3), generated by it, converges uniformly on the interval $[t_*, \vartheta]$ to a program motion $x^*(t) = x(t, t_*, x_*, \eta^*)$ from (2.3). However, by the construction of $x^*(t)$, we now have

$$\rho(x^*(\vartheta), M) = \lim_{j\to\infty} \rho(x^{(k_j)}(\vartheta), M) \tag{3.7}$$

Relations (3.6) and (3.7) are contradictory. The contradiction obtained proves that the program $\{\eta, [t_*, \vartheta)\}_{\Pi}^*$ constructed is the desired maximizing program $\{\eta, [t_*, \vartheta), x_*\}_{\Pi}^{\circ}$, solving Problem 3.2. Thus, we have verified the existence of a solution of Problem 3.2 for every choice of position $\{t_*, x_*\}$ and of number $\vartheta > t_*$.

Generally speaking, a solution of Problem 3.2 can furnish a nonunique maximizing program $\{\eta, [t_*, \vartheta), x_*\}_{\Pi}^{\circ}$. Therefore, it turns out to be appropriate to introduce the notion of a maximal maximizing program $\{\eta(dt, du, dv), [t_*, \vartheta), x_*\}_{\Pi}^{\circ\circ}$ for Problem 3.2, as the program which contains every other maximizing program $\{\eta(dt, du, dv), [t_*, \vartheta), x_*\}_{\Pi}^{\circ}$ for this same problem. This program exists because as such, it is sufficient to select the weakly-closed union of all maximizing programs $\{\eta(dt, du, dv), [t_*, \vartheta), x_*\}_{\Pi}^{\circ}$. For the following it is important that with a change of initial point x_* the maximal maximizing program $\{\eta, [t_*, \vartheta), x_*\}_{\Pi}^{\circ\circ}$ varies weakly upper-semicontinuously with respect to inclusion. Namely, the following assertion is valid.

L e m m a 3.1. Let $\{x^{(k)}\}$ and $\eta^{(k)} \in \{\eta, [t_*, \vartheta), x^{(k)}\}_{\Pi}^{\circ\circ}$ be sequences of initial points and of admissible controls, moreover, let $\lim x^{(k)} = x_*$ and let the measures

$\eta^{(k)}(dt, du, dv)$ converge weakly to the measure $\eta^*(dt, du, dv)$ as $k \to \infty$. Then, the admissible program control $\eta^* = \eta^*(dt, du, dv)$ is contained in the maximal maximizing program $\{\eta, [t_*, \vartheta), x_*\}_\Pi^{\circ\circ}$.

For the proof we form a weakly-closed set $\{\eta\}^*$ composed of all possible weak limits η for all possible weakly-convergent subsequences of admissible controls $\eta \in \{\eta, [t_*, \vartheta), x^{(k_j)}\}_\Pi^{\circ\circ}$. We can again verify that the set $\{\eta\}^*$ forms an admissible program. Furthermore, this program $\{\eta, [t_*, \vartheta), x_*\}_\Pi^*$ is the maximizing program $\{\eta, [t_*, \vartheta), x_*\}_\Pi^c$. Indeed, otherwise we would find a weakly-convergent subsequence $\eta_{(k_j)}(dt, du, dv)$ weakly tending to $\eta_*(dt, du, dv)$ for which the sequence of program motions $x^{(k_j)}(t) = x(t, t_*, x^{(k_j)}, \eta_{(k_j)})$ converges uniformly to the motion $x^*(t) = x(t, t_*, x_*, \eta_*)$ such that

$$\lim_{j \to \infty} \rho(x^{(k_j)}(\vartheta), M) = \rho(x^*(\vartheta), M) < \min_{\eta \in \{\eta\}_\Pi^{\circ\circ}} \rho(x(\vartheta, t_*, x_*, \eta), M) \quad (3.8)$$

But then, as a consequence of the uniform and equicontinuous dependence of the solutions $x(t)$ of integral equation (2.3) on the initial data, we would obtain that for sufficiently large values of j,

$$\min_{\eta \in \{\eta, [t_*, \vartheta), x_*\}_\Pi^{\circ}} \rho(x(\vartheta, t_*, x^{(k_j)}, \eta), M) >$$

$$\min_{\eta \in \{\eta, [t_*, \vartheta), x^{(k_j)}\}_\Pi^{\circ}} \rho(x(\vartheta, t_*, x^{(k_j)}, \eta), M) \quad (3.9)$$

However, inequality (3.9) contradicts the assumption that $\{\eta, [t_*, \vartheta), x^{(k_j)}\}_\Pi^{\circ}$ is a maximizing program for the initial position $\{t_*, x^{(k_j)}\}$. The contradiction obtained shows that $\{\eta, [t_*, \vartheta), x_*\}_\Pi^* = \{\eta, [t_*, \vartheta, x_*\}_\Pi^{\circ}$, i.e. $\{\eta, [t_*, \vartheta), x_*\}_\Pi^* \subset \{\eta, [t_*, \vartheta), x_*\}_\Pi^{\circ\circ}$ and, therefore, the weak limit $\eta^*(dt, du, dv)$ for every sequence $\{\eta^{(k)}(dt, du, dv)\}$ $(k = 1, 2, \ldots)$ of the form being considered is, indeed, contained in the program $\{\eta, [t_*, \vartheta), x_*\}_\Pi^{\circ\circ}$. By the same token, we have proven Lemma 3.1.

4. Regularity conditions. By the symbol $\varepsilon_0(t_*, x_*, \vartheta)$ we denote the quantity on the right-hand side of equality (3.4), i.e.,

$$\varepsilon_0(t_*, x_*, \vartheta) = \max_{\{\eta\}_\Pi} \min_{\eta \in \{\eta\}_\Pi} \rho(x(\vartheta, t_*, x_*, \eta), M) \quad (4.1)$$

We choose a certain position $\{t_*, x_*\}$ for which

$$\varepsilon_0(t_*, x_*, \vartheta) = \varepsilon > 0 \quad (4.2)$$

Let $t^* = t_* + \delta \leqslant \vartheta$. We fix some admissible extremal program $\{\eta(dt, du, dv), [t_*, t^*), x_*, s\}_\Pi$. The admissible program controls $\eta(dt, du, dv)(t_* \leqslant t < t^*)$ from the extremal program selected, generate in accordance with Eq. (2.3), a certain set of program motions $x(t) = x(t, t_*, x_*, \eta)(t_* \leqslant t \leqslant t^*)$ whose final values $x = x(t^*)$ constitute a certain set $X(t^*, \{t_*, x_*\}, \{\eta\}_\Pi)$. Along with the program motions $x(t)$ which are defined by Eq. (2.3), we consider also certain auxiliary motions $x(t)^* = x(t, t_*, x_*, \eta)^*$ $(t_* \leqslant t \leqslant t^*)$ which are defined by the equalities

$$x(t)^* = x_* + \int_{t_*}^{t} \int_P \int_Q f(\tau, x_*, u, v)\, \eta(d\tau, du, dv) \quad (4.3)$$

The final values $x = x(t^*)^*$ of these motions form a certain set $X^*(t^*, \{t_*, x_*\},$

$\{\eta\}_{\text{II}}$). It is not difficult to verify that under one and the same admissible control $\eta = \eta\,(dt,\,du,\,dv)$ the motions $x\,(t)$ and $x\,(t)^*$ satisfy the estimate

$$\| x\,(t^*)^* - x\,(t^*)\| \leqslant \varphi\,(\delta)\,\delta \tag{4.4}$$

where the function $\varphi\,(\delta)$ satisfies the condition

$$\lim_{\delta\to 0} \varphi\,(\delta) = 0 \tag{4.5}$$

and, so, forms an infinitesimal $o\,(\delta) = \varphi\,(\delta)\,\delta$ of a higher order of smallness relative to $\delta > 0$. Here, the estimate (4.4) is uniform in η in every bounded closed region G of space $\{x\}$ for $t_0 \leqslant t_* \leqslant \vartheta$. Thus, the distance $\gamma\,(X^*,\,X)$ between sets X^* and X, defined as the quantity

$$\gamma\,(X^*, X) = \max\,[\max_{x\in X} \rho\,(x, X^*) \quad \max_{x^*\in X^*} \rho\,(x, X^*)] \tag{4.6}$$

satisfies the estimate

$$\gamma\,(X^*,\ X) \leqslant \varphi\,(\delta)\,\delta \tag{4.7}$$

It is important to note that by the properties of the extremal program $\{\eta,\,[t_*,\,t^*),\,x_*,\,s\}_{\text{II}}$ the set X^* is bounded, convex, and closed.

We now select some point $x^* \in X^*$. The position $\{t^*,\,x^*\}$ corresponds to the maximal maximizing program $\{\eta,\,[t^*,\vartheta),\,x^*\}^{\circ\circ}_{\text{II}}$. By the choice of the initial position $\{t_*,\,x_*\}$ (4.2) and by definition of the quantity $\varepsilon_0\,(t_*,\,x_*,\,\vartheta) = \varepsilon$ (4.1) and of the set X, we conclude that we can find a control $\eta\,(dt,du,dv) \in \{\eta\,[t^*,\vartheta),\,x^*\}^{\circ\circ}_{\text{II}}$ and a point $x \in X$ such that the condition

$$\rho\,(x\,(\vartheta),\ M) \leqslant \varepsilon \tag{4.8}$$

is fulfilled for the corresponding program motion $x\,(t) = x\,(t,\ t^*,\ x,\ \eta)$ $(t^* \leqslant t \leqslant \vartheta)$ (2.3). Hence, as a consequence of (4.7) we derive that in set X^* we can find a point $x = x^{**}$ and a control $\eta\,(dt,\,du,\,dv) \in \{\eta,\,[t^*,\vartheta),\,x^*\}^{\circ\circ}_{\text{II}}$ such that the condition

$$\rho\,(x),\vartheta),\ M) \leqslant \varepsilon_* = \varepsilon + K\varphi\,(\delta)\,\delta \tag{4.9}$$

is fulfilled for the corresponding program motion $x\,(t) = x\,(t,\ t^*,\ x^{**}\ \eta)$ $(t^* \leqslant t \leqslant \vartheta)$ 2.3), where the constant $K = \exp\lambda\,(\vartheta - t_0)$ and λ is the Lipschitz constant for the right-hand side f of Eq. (1.1) in a suitable sufficiently-large bounded region G of space $\{x\}$ in which lie all the motions being considered. Inequality (4.9) is derived as a consequence of the inequality

$$\|x\,(t, t^*, x^*, \eta) - x\,(t, t^*, x, \eta)\| \leqslant \|x^* - x\|\exp\lambda\,(t - t^*) \tag{4.10}$$

which relates every two solutions $x\,(t,\,t^*,\,x^*,\,\eta)$ and $x\,(t,\,t^*,\,x,\,\eta)$ of Eq. (2.3) under one and the same program control η.

Thus for each point $x^* \in X^*$ we can construct a certain nonempty set $Y\,(x^*)$ consisting of all points $x^{**} \in X^*$ for each of which we can find at least one admissible control $\eta \in \{\eta,\,[t^*,\vartheta),\,x^*\}^{\circ\circ}_{\text{II}}$ such that the corresponding program motion $x\,(t) = x\,(t,\ t^*,\ x^{**},\ \eta)$ $(t^* \leqslant t \leqslant \vartheta)$ (2.3) satisfies condition (4.9). By the symbol $Y^*\,(x^*)$ we denote the closed convex hull of set $Y\,(x^*)$. We can now formulate one of the appropriate conditions for the regularity of the game.

C o n d i t i o n 4.1. We say that the game is regular for some value $\vartheta > t_0$ if for any sufficiently small value of $\beta > 0$ and any bounded closed region G in space

$\{x\}$ we can find a function $\varphi^*(\delta)$ satisfying the condition

$$\lim_{\delta\to 0} \varphi^*(\delta) = 0 \tag{4.11}$$

and such that for any position $\{t_*, x_*\}$, $x_* \in G$, $t_0 \leqslant t_* \leqslant \vartheta$, $\varepsilon_0(t, x_*, \vartheta) = \varepsilon \in [\beta, \beta^\circ]$, for any extremal program $\{\eta, [t_*, t^*), x_*, s\}_\Pi$, and for any point $x^* \in X^*(t^*, \{t_*, x_*\}, \{\eta\}_\Pi)$, we can find, for every point $x \in Y^*(x^*)$, a control $\eta(dt, du, dv) \in \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ}$ such that the condition

$$\rho(x(\vartheta), M) \leqslant \varepsilon^* = \varepsilon + \varphi^*(\delta)\,\delta \tag{4.12}$$

is fulfilled for the corresponding program motion $x(t) = x(t, t^*, x, \eta)$ $(t^* \leqslant t \leqslant \vartheta)$ (2.3).

Here and below β° denotes a certain sufficiently-small fixed positive number and $t^* = t_* + \delta$. Note that for the fulfillment of Condition 4.1 it is obviously enough that the sets $Y(x^*)$ themselves prove to be convex sets. This condition turns out to be a natural one when Eq. (1.1) is linear, even if with respect to x, and the target set turns out to be convex (for example, see [1, 6]). Moreover, for the fulfillment of Condition 4.1 it is sufficient that the sets $Y(x^*)$ and $Y^*(x^*)$ satisfy the estimate

$$\gamma(Y(x^*), Y^*(x^*)) \leqslant \varphi_*(\delta)\,\delta \tag{4.13}$$

where

$$\lim_{\delta\to 0} \varphi_*(\delta) = 0 \tag{4.14}$$

because to fulfill condition (4.2) it is sufficient to set $\varphi^*(\delta) = K(\varphi(\delta) + \varphi_*(\delta))$.

Now by the symbol $Y_{\min}(x^*)$ we denote the set of those points $x^\circ \in X^*(t^*, \{t_*, x_*\}, \{\eta\}_\Pi)$, for which the condition

$$\min_{\eta\in\{\eta\}_\pi^{\circ\circ}} \rho(x(\vartheta, t^*, x^\circ, \eta), M) = \min_{x\in X^*} \min_{\eta\in\{\eta\}_\pi^{\circ\circ}} \rho(x(\vartheta, t^*, x, \eta), M) \tag{4.15}$$

is fulfilled, where the minimum is taken over all controls $\eta \in \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ}$. By the symbol $Y^*_{\min}$ we denote the closed convex hull of set $Y_{\min}$. Then, another appropriate regularity condition can be stated in the following way.

C o n d i t i o n 4.2. We say that the game is regular for some value $\vartheta > t_0$ if for any sufficiently small value of $\beta > 0$ and for any bounded closed region G in space $\{x\}$ we can find a function $\varphi_*(\delta)$ satisfying condition (4.14) and such that

$$\gamma(Y_{\min}(x^*), Y^*_{\min}(x^*)) \leqslant \varphi_*(\delta)\,\delta \tag{4.16}$$

for any position $\{t_*, x_*\}$, $x_* \in G$, $t_0 \leqslant t_* \leqslant \vartheta$, $\varepsilon_0(t_*, x_*, \vartheta) = \varepsilon \in [\beta, \beta^\circ]$, for any extremal program $\{\eta, [t_*, t^*), x_*, s\}_\Pi$, and for any point $x^* \in X^*(t^*, \{t_*, x_*\}, \{\eta\}_\Pi)$, and if the maximal maximizing programs $\{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ}$ are weakly continuous in x^* for $\varepsilon_0(t^*, x^*, \vartheta) \in (0, \beta^\circ]$.

For the fulfillment of Condition 4.2 it is obviously sufficient that along with the condition of weak continuity of the programs $\{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ}$ in x^* there should be fulfilled the convexity condition for the sets $Y_{\min}(x^*)$ themselves, in particular, the condition for the uniqueness of the point $\overset{\circ}{x}_{\min} \in X^*(x^*)$ satisfying condition (4.15).

5. Program absorption sets. Let us fix a certain value $\vartheta > t_0$. Let us divide the entire halfspace $\{t, x\}$, $t \leqslant \vartheta$ into two parts. To the first part we refer the region wherein the inequality $\varepsilon_0(t, x, \vartheta) > 0$ is satisfied. To the second part we refer the set of positions $\{t, x\}$ for which $\varepsilon_0(t, x, \vartheta) = 0$. The quantity $\varepsilon_0(t, x, \vartheta)$ is a continuous function of position $\{t, x\}$. This assertion derives as a consequence of the continuous dependence of the solutions $x(t)$ of Eq. (2.3) on the initial data by arguments similar to those presented in Sect. 3 to prove the weak upper-semicontinuity relative to inclusion of the maximal maximizing programs with respect to a change in the initial point x_*. We omit this proof here.

But from the continuity of the function $\varepsilon_0(t, x, \vartheta)$ with respect to position $\{t, x\}$ it follows that the region $\varepsilon_0(t, x, \vartheta) > 0$, $t < \vartheta$ is open in the space of $\{t, x\}$. At the same time, any set of positions $\{t, x\}$, where $t_0 \leqslant t \leqslant \vartheta$ and $\varepsilon_0(t, x) \leqslant \varepsilon$ or $\varepsilon_0(t, x) \geqslant \varepsilon$ ($\varepsilon \leqslant 0$ is a constant), is the closed set. In particular, we are especially interested in the closed set W_0 of positions $\{t, x\}$ for which $\varepsilon_0(t, x, \vartheta) = 0$, $t_0 \leqslant t \leqslant \vartheta$. The following condition is obviously fulfilled for this set which we call the minimax program absorption set. Whatever the position $\{t_*, x_*\} \in W_0$ and no matter what the second player's program $\{\eta, [t_*, \vartheta)\}_\Pi$ turns out to be, we can always find at least one admissible control η from this program, such that the condition $x(\vartheta) \in M$ is fulfilled for the program motion $x(t) = x(t, t_*, x_*, \eta)$ generated by it. Further, by the symbol W_ε ($\varepsilon \geqslant 0$) we denote the closed set of positions $\{t, x\}$ for which the condition $\varepsilon_0(t, x, \vartheta) \leqslant \varepsilon$, $t_0 \leqslant t \leqslant \vartheta$ is satisfied. The following assertion is valid.

Lemma 5.1. Suppose that for a chosen value $\vartheta > t_0$ the game is regular in the sense of Condition 4.1 and that a bounded region G in space $\{x\}$ has been chosen. Then, whatever the value of $\varepsilon \in [\beta, \beta^\circ]$, whatever be the position $\{t_*, x_*\}$ ($t_* \leqslant \vartheta$), for which $x_* \in G$ and $\varepsilon_0(t_*, x_*, \vartheta) = \varepsilon$, and whatever be the extremal program $\{\eta, [t_*, t^*), x_*, s\}_\Pi$ ($t^* - t_* = \delta$, $t^* \leqslant \vartheta$), among the controls $\eta = \eta(dt, du, dv)$ contained in this program, we can find at least one control $\eta^* = \eta^*(dt, du, dv)$ which generates the auxiliary motion $x(t)^* = x(t, t_*, x_*, \eta^*)^*$ ($t_* \leqslant t \leqslant t^*$) (4.3) satisfying the condition

$$\{t^*, x(t^*)^*\} \in W_{\varepsilon^*} \tag{5.1}$$

where $\varepsilon^* = \varepsilon + \varphi^*(\delta)\delta$.

Let us prove the lemma. We assume to the contrary that the lemma is incorrect. Then we can find a number $\varepsilon \in [\beta, \beta^\circ]$, a position $\{t_*, x_*\}$, $x_* \in G$, $t_* < \vartheta$ for which $\varepsilon_0(t_*, x, \vartheta) = \varepsilon$, and an extremal program $\{\eta, [t_*, t^*), x_*, s\}_\Pi$ ($t^* - t_* = \delta$, $t^* \leqslant \vartheta$), such that all auxiliary motions $x(t) = x(t, t_*, x_*, \eta)^*$ ($t_* \leqslant t \leqslant t^*$, $\eta \in \{\eta\}_\Pi$) 4.3) satisfy the condition

$$\{t^*, x(t^*)^*\} \notin W_{\varepsilon^*} \tag{5.2}$$

We consider the mapping $x^* \to Y^*(x^*)$ of elements $x^* \in X^*$ onto the subsets $Y^*(x^*) \subset X^*$. This mapping leads to a contradiction. Under the assumption (5.2) made, no element $x^* \in X^*$ can be contained in its own image $Y^*(x^*)$. Indeed, let us assume to the contrary that a certain element x^* is contained in its own image $Y^*(x^*)$. Then by regularity Condition 4.1 we can find a control $\eta \in \{\eta\ [t^*, \vartheta), x^*\}_\Pi^{\circ\circ}$ which generates a program motion $x(t) = x(t, t^*, x^*, \eta)$ ($t^* \leqslant t \leqslant \vartheta$) (2.3) satisfying the condition

$$\rho(x(\vartheta), M) \leqslant \varepsilon + \varphi^*(\delta)\,\delta = \varepsilon^* \tag{5.3}$$

In other words, if $x^* \in Y^*(x^*)$, this signifies that in the maximal maximizing program $\{\eta, [t^*, \vartheta), x_1^*\}_\Pi^{\circ\circ}$ we can find a control η which generates a program motion $x(t) = x(t, t^*, x^*, \eta)$ ($t^* \leqslant t \leqslant \vartheta$) (2.3) satisfying condition (5.3). But by the definition of the quantity $\varepsilon_0(t^*, x^*, \vartheta)$ this signifies the fulfillment of the inequality

$$\varepsilon_0(t^*, x^*, \vartheta) \leqslant \varepsilon^* \tag{5.4}$$

which, in its own turn, by the definition of set W_{ε^*} signifies the fulfillment of the inclusion

$$\{t^* x^*\} \in W_{\varepsilon^*} \tag{5.5}$$

But conditions (5.2) and (5.5) are contradictory, whence it follows that there is no element $x^* \in X^*$ whatsoever which can be contained in its own image $Y^*(x^*) \subset X^*$

However, the mapping $x^* \to Y^*(x^*)$ is a mapping of elements x^* of a convex bounded and closed set X^* onto bounded, convex and closed subsets $Y^*(x^*) \subset X^*$ The sets $Y^*(x^*)$ turn out to be upper-semicontinuous relative to inclusion with respect to a change in element x^* Let us show this. We take a sequence of points $x^{(k)}$ $(k = 1, 2, \ldots)$ converging to point x^* as $k \to \infty$. It is known that the maximal maximizing programs $\{\eta, [t^*, \vartheta), x\}_\Pi^{\circ\circ}$ are weakly upper-semicontinuous relative to inclusion with respect to a change in point x. Hence it follows that as $k \to \infty$ the sets $Y(x^{(k)})$ converge to set $Y(x^*)$ upper-semicontinuously relative to inclusion. Indeed, let us accept to the contrary that some convergent subsequence of points $x^{[j]} \in Y(x^{(k_j)})$ has, as $j \to \infty$, a limit point $x^{[*]}$ which is not contained in the set $Y(x^*)$. But, by the definition of the sets $Y(x^{(k_j)})$, for each point $x^{(k_j)}$ we can find a control $\eta^{(k_j)} \in \{\eta, [t^*, \vartheta), x^{(k_j)}\}_\Pi^{\circ\circ}$ which generates a program motion $x(t) = x(t, t^*, x^{(k_j)}, \eta^{(k_j)})$ $(t^* \leqslant t \leqslant \vartheta)$ (2.3) satisfying the condition

$$\rho(x(\vartheta), M) \leqslant \varepsilon + \lambda\varphi(\delta)\delta \tag{5.6}$$

The sequence $\eta^{(k_j)}$ $(j = 1, 2, \ldots)$ can be assumed weakly convergent. The weak limit $\eta^{[*]}$ of this sequence should belong to the maximal maximizing program $\{\eta, [t^*, \vartheta), x^{[*]}\}_\Pi^{\circ\circ}$. But

$$\lim_{j\to\infty} x(t, t^*, x^{(k_j)}, \eta^{(k_j)}) = x(t, t^*, x^{[*]}, \eta^{[*]}) \tag{5.7}$$

therefore, condition (5.6) is satisfied also for the limit program motion $x(t) = x(t, t^*, x^{[*]}, \eta^{[*]})$. But this means that $x^{[*]} \in Y(x^*)$. The contradiction obtained proves the upper-semicontinuity relative to inclusion of the sets $Y(x^*)$ with respect to a change in x^*. Then, the closed convex hulls $Y^*(x^*)$ of the sets $Y(x^*)$ also possess this property.

The mapping $x^* \to Y^*(x^*)$ now satisfies all the hypotheses of the theorem in [7]. According to this theorem the mapping constructed has at least one fixed point x^*, i.e. there exists an element $x^* \in X^*$ which satiafies the condition $x^* \in Y^*(x^*)$. But, as we have noted at the start of the proof of Lemma 5.1, this inclusion is impossible. The contradiction obtained proves the lemma. By analogous arguments we can also prove a Lemma 5.2, analogous to Lemma 5.1 but now starting not from regularity Condition 4.1 but from regularity Condition 4.2. Namely, the following assertion is valid.

Lemma 5.2. Suppose that for a chosen value $\vartheta > t_0$ the game is regular in the sense of Condition 4.2. Then, whatsoever be the bounded region G in space $\{x\}$ and the value $\varepsilon \in [\beta, \beta']$, whatever be the position $\{t_*, x_*\}$, $x_* \in G$, $t_0 \leqslant t_* \leqslant \vartheta$, for which $\varepsilon_0(t_*, x_*, \vartheta) = \varepsilon$, and whatever be the extremal program $\{\eta, [t_*, t^*), x_*, s\}_\Pi$ $(t^* - t_* = \delta, t^* \leqslant \vartheta)$, among the controls η contained in this extremal program we can find at least one control η^* which generates an auxiliary motion $x(t)^* = x(t, t_*, x_*, \eta^*)$ $(t_* \leqslant t \leqslant t^*)$ (4.3), satisfying condition (5.1).

The proof of Lemma 5.2, similar to the proof of Lemma 5.1 and differing from it only at that point where the weak upper-semicontinuity relative to inclusion of the mapping $x^* \to Y(x^*)$ is proved, is omitted.

6. Strong stability of set W_ε. Relying on Lemmas 5.1 and 5.2 let us prove the property, important for what is to follow, of minimax strong u-stability of set W_ε for $\varepsilon \in (0, \beta^\circ)$. This property is formulated in the following way. We say that a certain set W in the half-space $\{t, x\}$, $t \leqslant \vartheta$ is minimax strongly u-stable if, whatever be the position $\{t_*, x_*\} \in W$ $(t_* \leqslant \vartheta)$, the number $t^* \in (t_*, \vartheta]$ and the function V which associates a set $V(u) \subset Q$ with every vector $u \in P$, among the solutions $x(t) = x(t, t_*, x_*, V)$ of the contingent equation

$$x^{\cdot}(t) \in F_V(t, x(t)) \tag{6.1}$$

we can find at least one solution $x(t)$ satisfying the condition $\{t^*, x(t^*)\} \in W$. Here the symbol $F_V(t, x)$ denotes the closed convex hull of the set of vectors $f = f(t, x, u, v)$, which is obtained when the vector v ranges over $V(u)$ and the vector u ranges over the whole set P. The following assertion is valid.

Lemma 6.1. Suppose that for a chosen value ϑ the game is regular in the sense of Condition 4.1 or 4.2. Then, the sets W_ε are minimax strongly u-stable for every $\varepsilon \in (0, \beta^\circ)$

To prove Lemma 6.1 we construct that motion $x(t) = x(t, t_*, x_*, V)$ (6.1) which satisfies the condition stated in the minimax strong u-stability property of set W_ε. We select some value of $\varepsilon \in (0, \beta^\circ)$. Further, suppose that a position $\{t_*, x_*\} \in W_\varepsilon$, an instant $t^* \leqslant \vartheta$ and the function $V(u)$ appearing in the stability conditions, have been chosen. We construct a set $W_\varepsilon^{(\delta)}$ in the space $\{t, x\}$ composed from the sets $W_{\varepsilon+\varphi^*(\delta)(t-t_*),t}$ $(t_* \leqslant t \leqslant t^*)$ which are sections of the sets $W_{\varepsilon+\varphi^*(\delta)(t-t_*)}$ by the hyperplanes $t = \text{const}$. Here $\varphi^*(\delta)$ is the function appearing in Lemmas 5.1 and 5.2.

Let us consider Euler's polygonal line $x_{\Delta(\delta)}[t] = x_{\Delta(\delta)}[t, t_*, x_*, U_\varepsilon^{(\delta)}, v[\cdot]]$ (1.4), where $v^*[t] = v_i^* \in V(u_i)$, $\tau_i \leqslant t < \tau_{i+1}$ $(\tau_0 = t_*, \tau_{i+1} = \tau_i + \delta)$ and the control $u_i = u(\tau_i, x_{\Delta(\delta)}[\tau_i])$ $(\tau_i \leqslant t < \tau_{i+1})$ is determined by the strategy $U_\varepsilon^{(\delta)}$ extremal [2] to set $W_\varepsilon^{(\delta)}$. This strategy determines the function $u(t, x)$ for $t_* \leqslant t \leqslant t^*$ in the following manner. If $\{t, x\} \in W_\varepsilon^{(\delta)}$, then as $u(t, x)$ we can choose any vector $u \in P$. However, if $\{t, x\} \notin W_\varepsilon^{(\delta)}$, then as $u(t, x)$ we can choose any vector $u_e \in P$ satisfying the condition

$$\max_{v \in Q} s' f(t, x, u_e, v) = \min_{u \in P} \max_{v \in Q} s' f(t, x, u, v) \tag{6.2}$$

Here s is the vector $x - x_w$, where x_w is a point from set $W_{\varepsilon,t}^{(\delta)}$, which us closest to point x in the sense of the Euclidean metric. The following estimate is valid for the Euler's polygonal line $x_{\Delta(\delta)}[t]$ being considered. Suppose that a certain position $\{\tau_i, x_{\Delta(\delta)}[\tau_i]\} \notin W_\varepsilon^{(\delta)}$ has been realized on this polygonal line $x_{\Delta(\delta)}[t]$ at some instant $t = \tau_i$. Then

$$\rho^2(x_{\Delta(\delta)}[\tau_{i+1}], W_{\varepsilon+\varphi^*(\delta)(\tau_{i+1}-t_*),\tau_{i+1}}) \leqslant \rho^2(x_{\Delta(\delta)}[\tau_i], W_{\varepsilon+\varphi^*(\delta)(\tau_i-t_*),\tau_i})(1 + 2\lambda\delta) + o(\delta) \tag{6.3}$$

where $o(\delta)$ denotes a higher-order infinitesimal relative to δ, while λ is the Lipschitz constant for the right-hand side of Eq. (1.1) in that region G of space $\{x\}$, wherein lie all the motions being considered. The estimate (6.3) is uniform for all considered values of τ_i along all the Euler's polygonal lines $x_{\Delta(\delta)}[t]$ of interest to us.

Estimate (6.3) is derived by means of comparing the segment of interest of the Euler's polygonal line $x_{\Delta(\delta)}[t]$ $(\tau_i \leqslant t \leqslant \tau_{i+1})$ with a certain program motion $x(t)^* = x(t, \tau_i, x_w^{(i)}, \eta)^*$, $\eta \in \{\eta, [\tau_i, \tau_{i+1}), x[\tau_i], x[\tau_i] - x_w^{(i)}\}_\Pi$, where $x_w^{(i)}$ is a point from $W^{(\delta)}_{\varepsilon, \tau_i}$, nearest to $x[\tau_i]$. Here the motion $x(t)^*$ indicated is precisely that program motion which in accordance with Lemma 5.1 or with Lemma 5.2 satisfies the condition

$$\{\tau_{i+1}, x(\tau_{i+1})^*\} \in W^{(\delta)}_{\varepsilon, \tau_{i+1}} \tag{6.4}$$

This comparison, leading to estimate (6.3), is omitted here since it is analogous to the arguments in [2], only that the saddle-point condition (see condition (2.1) in [2]) occurring therein should be replaced by the minimax condition (6.2) occurring herein, taking into account condition (2.7) characterizing the extremal program $\{\eta, [\tau_i, \tau_{i+1}), x[\tau_i], x[\tau_i] - x_w^{(i)}\}_\Pi$.

The estimate

$$\rho^2 (x_{\Delta(\delta)}[t^*], W^{(\delta)}_{\varepsilon, t^*}) \leqslant O(\delta) \tag{6.5}$$

where $O(\delta)$ is a quantity satisfying the condition $\lim O(\delta) = 0$ as $\delta \to 0$, is derived from the estimate (6.3) for the whole of Euler's polygonal line $x_{\Delta(\delta)}[t]$ $(t_* \leqslant t \leqslant t^*)$ also by arguments analogous to those in [2]. It now remains to choose the sequence of numbers $\delta_k \to 0$ as $k \to \infty$ and to examine the sequence of corresponding Euler's polygonal lines $x^{(k)}[t] = x_\Delta(\delta_k)[t]$. From this sequence we can select a subsequence which converges uniformly on the interval $[t_*, t^*]$ to some absolutely continuous function $x^*(t) = x^*(t, t_*, x_*)$. From estimate (6.5) it follows that the limit function $x^*(t)$ satisfies the condition $\{t^*, x^*(t^*)\} \in W_\varepsilon$. On the other hand, we can verify (see [8]) that the limit function $x^*(t)$ constructed is a solution of the contingent equation (6.1). Thus, we have indeed constructed a solution $x^*(t) = x^*(t, t_*, x_*, V)$ of the contingent equation (6.1) satisfying the condition $\{t^*, x^*(t^*)\} \in W_\varepsilon$. By the same token we have proved Lemma 6.1.

7. Strong stability of the absorption set W_0. The minimax strong u-stability of sets W_ε for $\varepsilon \in (0, \beta^0)$, proven in Sect. 6, permits us to establish that the absorption set W_0 also possesses the analogous property. Let us show this. The following assertion is valid.

Lemma 7.1. Suppose that for a chosen value $\vartheta > t_0$ the game is regular in the sense of Condition 4.1 and 4.2. Then the minimax absorption set W_0 is a minimax strongly u-stable set.

Indeed, let us assume that Lemma 7.1 is incorrect. Then we can find a function $V(u)$, a position $\{t_*, x_*\} \in W_0$ and an instant $t^* \in (t_*, \vartheta]$, such that all solutions $x(t) = x(t, t_*, x_*, V)$ of Eq. (6.1) satisfy the condition

$$\{t^*, x(t^*)\} \notin W_0 \tag{7.1}$$

Since set W_0 and the set of all points $\{t^*, x(t^*)\}$ are closed sets, then from (7.1) follows the inequality

$$\rho(\{t^*, x(t^*)\}, W_0) \geqslant \nu > 0 \tag{7.2}$$

and, therefore, as a consequence of the continuity in t and x of the function $\varepsilon_0(t, x, \vartheta)$ we have the inequality

$$\varepsilon_0(t^*, x(t^*), \vartheta) \geqslant \varepsilon_\nu > 0 \tag{7.3}$$

where ε_ν is some positive number. Consequently, the relation

$$\{t^*, x(t^*)\} \notin W_{\varepsilon_\nu} \tag{7.4}$$

is valid for all the motions $x(t)$ being considered. But condition (7.4) contradicts the minimax strong u-stability property of set W_{ε_ν}, because $\{t_*, x_*\} \in W_0 \subset W_{\varepsilon_\nu}$. The contradiction obtained proves Lemma 7.1.

8. The basic result. The following assertion is valid.

Theorem 8.1. Suppose that for some value $\vartheta > t_0$ the initial position $\{t_0, x_0\} \in W_0$ and that the game is regular in the sense of Condition 4.1 or 4.2. Then, the strategy $U^\circ \div u^\circ(t, x)$, extremal to the set W_0, solves Problem 1.1. Here the inclusion

$$x[\vartheta] \in M \tag{8.1}$$

is valid for every motion $x[t] = x[t, t_0, x_0, U^\circ]$

Recall that the strategy $U^\circ \div u^\circ(t, x)$, extremal to set W_0, is defined by the following conditions. If position $\{t, x\} \in W_0$, then as $u^\circ(t, x)$ we can choose any vector $u \in P$. If, however, position $\{t, x\} \notin W_0$, then we construct the vector $s = x - x^\circ$ where x° is a point of the section $W_{0,t}$ of set W_0 by the hyperplane $t = \text{const}$, closest to point x in the Euclidean metric. Now as $u^\circ(t, x)$ we can choose any vector $u^\circ \in P$ which satisfies the condition

$$\max_{v \in Q} s' f(t, x, u^\circ, v) = \min_{u \in P} \max_{v \in Q} s' f(t, x, u, v) \tag{8.2}$$

The validity of Theorem 8.1 is derived immediately from Lemma 7.1 by arguments analogous to those presented in [2]. According to Theorem 8.1 regularity Conditions 4.1 and 4.2 prove to be sufficient for constructing the strategy $U^\circ \div u^\circ(t, x)$ which solves Problem 1.1 and consequently permits the first player to terminate the game successfully. These conditions cover many of those applicable in various encounter games (for example, see [1, 6, 9]), and first of all in pursuit games in the cases when the controls u and v of the right-hand side of Eq. (1.1) are additively partitioned. One sufficient condition for the regularity of a game in the sense of Condition 4.1 is given in Sect. 9.

9. Regular case. We present one case when the game is regular in the sense of Condition 4.1. We assume that the function $f(t, x, u, v)$ in the right-hand side of Eq. (1.1) has continuous partial derivatives $\partial f / \partial x_i$ $(i = 1, 2, \ldots, n)$. Then it is useful to state the following condition.

Condition 9.1. We say that the game is strongly regular for some value $\vartheta > t_0$ if the function $f(t, x, u, v)$ has continuous partial derivatives and if for every initial position $\{t_*, x_*\}$ $(t_* < \vartheta)$ satisfying the condition $\varepsilon_0(t_*, x_*, \vartheta) \in (0, 3\beta^\circ)$, Problem 3.2 has a unique solution $\eta^\circ(dt, du, dv \mid [t_*, \vartheta), x_*)^\circ$ and the point $x_M \in M$, closest to the point $x = x^\circ(\vartheta)^\circ = x^\circ(\vartheta, t_*, x_*, \eta^\circ \mid \vartheta)^\circ$, is unique.

The following assertion is valid.

Lemma 9.1. If Condition 9.1 is fulfilled for some value $\vartheta > t_0$, then the game is regular for this value ϑ in the sense of Condition 4.1.

Let us prove Lemma 9.1. We fix a certain position $\{t_*, x_*\}$ $(t_* < \vartheta)$, for which $\varepsilon_0(t_*, x_*, \vartheta) = \varepsilon \in [2\beta, \beta^\circ]$, $\beta > 0$, a number $t^* = t_* + \delta \leqslant \vartheta$ and a certain extremal program $\{\eta, [t_*, t^*), x_*, s\}_{\text{II}}$. As a consequence of the fact that the region $\varepsilon_0(t, x, \vartheta) > \beta \geqslant 0$ is open for $t < \vartheta$, we can assume the number $\delta > 0$ to be so small that all points $x = x(t^*, t_*, x_*, \eta)^*$, for $\eta \in \{\eta, [t_*, t^*), x_*, s\}_{\text{II}}$ remain in the region $\varepsilon_0(t^*, x, \vartheta) \in (\beta, 2\beta^\circ)$. Thus, we can assume the set X^* as lying in the region

$\varepsilon_0(t^*, x, \vartheta) \in (\beta, 2\beta^\circ)$. But then the condition $\varepsilon_0(t^*, x^*, \vartheta) \in (\beta, 2\beta^\circ)$ is fulfilled for any point $x^* \in X^*$ and, consequently, in accordance with Condition 9.1, Problem 3.2 has a unique solution $\eta^\circ(dt, du, dv \mid [t^*, \vartheta), x^*)^\circ$ for the position $\{t^*, x^*\}$, where $x^* \in X^*$.

We select some point $x^* \in X^*$. Let $x^\circ(t)^\circ = x^\circ(t, t^*, x^*, \eta^\circ \mid \vartheta)^\circ$ be the corresponding optimal program motion and let $x_M \in M$ be a point closest to the point $x = x^\circ(\vartheta)^\circ$. By the expression $G(t^*, x, \vartheta, \{\eta\}_\Pi)$ we denote the attainability region at the instant $t = \vartheta$ from the position $\{t^*, x\}$ for the program motions $x(t) = x(t, t^*, x, \eta)$ with $\eta \in \{\eta\}_\Pi$. In other words, $G(t^*, x, \vartheta, \{\eta\}_\Pi)$ is the set of points x in space $\{x\}$ over which the point $x(\vartheta) = x(\vartheta, t^*, x, \eta)$ ranges under all possible admissible program controls $\eta \in \{\eta, [t^*, \vartheta)\}_\Pi$.

Let us consider the region $G(t^*, x^*, \vartheta, \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ}$. By Condition 9.1 in this region there is a single point $x^\circ = x^\circ(\vartheta)^\circ$ closest to set M in which, in its own turn, there is a single point x_M closest to point x°; here, $\rho(x^\circ, M) = \varepsilon_0(t^*, x^*, \vartheta) \in (\beta, 2\beta^\circ)$. We now choose all possible points x from set X^*, however, leaving the program $\{\eta\}_\Pi = \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ}$ unaltered. To check Condition 4.1 we need to estimate here the set $Y^*(x^*)$ namely, the convex hull of the set $Y(x^*)$ of those points $x^{**} \in X^*$ for which the region $G(t^*, x^{**}, \vartheta, \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ})$ intersects with the ε_*-neighborhood of set M, where $\varepsilon_* = \varepsilon + K\varphi(\delta)\delta$ (4.9) and the function $\varphi(\delta)$ satisfies condition (4.5).

Our problem is to show that we can choose $\varepsilon^* = \varepsilon + \varphi^*(\delta)\delta$ (4.12) in such a way that for all points $x^{**} \in Y^*(x^*)$ each of the regions $G(t^*, x^{**}, \vartheta, \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ})$ intersects the ε^*-neighborhood of set M. To do this we consider the following geometric picture. By the symbol G_M we denote the closed $(-M)$-neighborhood of the attainability region G, i.e. the set of points $x - m$, where $x \in G$ and $m \in M$. Obviously, the region $G(t^*, x, \vartheta, \{\eta\}_\Pi)$ intersects some Euclidean ε-neighborhood of set M if and only if the corresponding region G_M intersects the sphere $\|x\| \leqslant \varepsilon$. Thus, we need to show that we can select $\varepsilon^* = \varepsilon + \varphi^*(\delta)\delta$ such that for all points $x^{**} \in Y^*(x^*)$ each of the regions $G_M(t^*, x^{**}, \vartheta, \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ})$ intersects the ε^*-sphere $\|x\| \leqslant \varepsilon^*$, and the function $\varphi^*(\delta)$ satisfies condition (4.11).

To do this we first estimate the set $Y_*(x^*)$ of those points $x_{**} \in X^*$ for which the point $x = x(\vartheta) = x(\vartheta, t^*, x_{**}, \eta^\circ(dt, du, dv \mid [t^*, \vartheta), x^*)^\circ) + s^\circ$ intersects with some suitable sphere $\|x\| \leqslant \varepsilon^*$. Here $s^\circ = x_M - x^\circ(\vartheta)^\circ$. Since the right-hand side $f(t, x, u, v)$ of Eq. (1.1) is a differentiable function, the change in the solution $x(t)$ of Eq. (2.3) under a variation of only the initial condition $\Delta x(t^*) = x - x^*$ with an unaltered program control $\eta = \eta^\circ$, is defined in the first approximation and to within terms of a higher order of smallness relative to Δx (and, therefore, by the choice of $x \in X^*$) and also to within terms of a higher order of smallness relative to δ, by the solution of the integral variational equation

$$\delta x(t) = \delta x(t^*) + \int_{t^*}^{t} \int_P \int_Q \Big[\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \delta x_i(\tau) \Big] \eta^\circ(d\tau, du, dv) \tag{9.1}$$

Thus, when the point x describes the set X^*, the point $x = x(\vartheta) + s^\circ$ describes, to within terms of order $o(\delta)$, a certain set X_ϑ^* obtained by means of some nonsingular linear transformation B of the set X^* This set $X_\vartheta^* = \{x^{(\vartheta)}\}$ is described by the relation

$$x^{(\vartheta)} = x^\circ(\vartheta)^\circ + s^\circ + B\delta x(t^*) = x^\circ(\vartheta)^\circ + s^\circ + \delta x(\vartheta) \quad (9.2)$$
$$\text{for} \quad \delta x(t^*) = x - x^*$$

Thus, as is the set X^*, the set X_ϑ^* also proves to be bounded, convex, and closed. But in this case the intersection Z_ϑ^* of set X_ϑ^* with any σ-sphere $\|x\| \leqslant \sigma$ is also a bounded convex closed set and its preimage Z^* under transformation (9.2) is as well a bounded convex closed set.

Let us now show that for a suitable choice of $\sigma(\delta) = \varepsilon^* = \varepsilon_0(t_*, x_*, \vartheta) + \varphi^*(\delta)\delta$ $(\varphi^*(\delta) \to 0$ as $\delta \to 0)$ the closed convex hull $Y^*(x^*)$ of the set $Y(x^*)$ is contained in set Z^*. This, by the same token, will prove Lemma 9.1. Thus, let us prove that the inclusion $Y^*(x^*) \subset Z^*$ is valid for a suitable choice of $\sigma = \varepsilon^*$. For this purpose, around the point $x = x^\circ(\vartheta)^\circ + s^\circ$ in the region $G_M(t^*, x^*, \vartheta, \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ})$ we isolate a certain $O(\delta)$-neighborhood $G_M^{(\delta)}$ which in this region and in the regions $G_M(t^*, x, \vartheta, \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ})$ deformed as a result of a change in the initial condition x can intersect only with the ε_*-sphere $\|x\| \leqslant \varepsilon_*$. Here, for different initial points x^* and x in the regions $G_M(t^*, x^*, \vartheta, \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ})$ and $G_M(t^*, x, \vartheta, \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ})$ we identify those points to which the corresponding motions $x(t) = x(t, t^*, x^*, \eta) + s\,\delta(t - \vartheta)$ and $x(t) = x(t, t^*, x, \eta) + s\delta(t - \vartheta)$ arrive at the instant $t = \vartheta$ under one and the same control $\eta \in \{\eta\}_\Pi^{\circ\circ}$ and for the same $s = -m$, $m \in M$. Here $O(\delta)$ denotes a quantity which satisfies the condition $\lim O(\delta) = 0$ as $\delta \to 0$ and $\delta(t)$ denotes the Dirac δ-function. We can verify that under a change of initial conditions $\Delta x = x - x^*$ the isolated piece $G_M^{(\delta)}$ of region G_M is displaced and is deformed such that to within a displacement of order $o(\delta)$ it is displaced translationally along the vector $B\Delta x$.

Now let some point $x^{**} \in Y(x^*)$. This signifies that in the region $G_M^{(\delta)}(t^*, x^{**}, \vartheta, \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ})$ there is a point $x_\vartheta^{**} = x(\vartheta, t^*, x^{**}, \eta^{**}) + s^{**}$ which lies in the ε_*-sphere $\|x\| \leqslant \varepsilon_*$. We can show that under this condition a certain point $x_{**}^{(\vartheta)} \in X_\vartheta^*$, representing the point $x(\vartheta, t^*, x^{**}, \eta^\circ) + s^\circ$ in the linear approximation (9.1), lies in an appropriate ε^*-sphere $\|x\| \leqslant \varepsilon^*$.

Let us show this. Suppose that the point $x(\vartheta, t^*, x^{**}, \eta^\circ) + s^\circ$ has been shifted relative to the point $x(\vartheta, t^*, x^*, \eta^\circ) + s^\circ$ by a certain vector $\Delta x(\vartheta)$. Then, to within a term of order $o(\delta)$, the point $x(\vartheta, t^*, x^{**}, \eta^{**}) + s^{**}$ is shifted relative to the point $x(\vartheta, t^*, x^*, \eta^{**}) + s^{**}$ by that same vector $\Delta x(\vartheta)$. Further, to within terms of order $o^*(\delta)$, the change in the distance from the point $x(\vartheta, t^*, x^*, \eta^\circ) + s^\circ$ to the point $x = 0$ under the displacement $\Delta x(\vartheta) = x(\vartheta, t^*, x^{**}, \eta^\circ) - x(\vartheta, t^*, x^*, \eta^\circ)$ is represented by the scalar product $[x(\vartheta, t^*, x^*, \eta^\circ) + s^\circ]'\,\Delta x(\vartheta) \,/\, \|x(\vartheta, t^*, x^*, \eta^\circ) + s^\circ\|$. It is easy to see that with the same accuracy this scalar product also represents the change in the distance from the point $x(\vartheta, t^*, x^{**}, \eta^{**}) + s^{**}$ to the point $x = 0$ under the displacement $\Delta x^{**}(\vartheta) = x(\vartheta, t^*, x^{**}, \eta^{**}) - x(\vartheta, t^*, x^*, \eta^{**})$. Here the estimates are uniform in every bounded closed region G of space $\{x\}$ for the values $\varepsilon_0(t_*, x_*, \vartheta) = \varepsilon \in (2\beta, \beta^\circ)$. Since the quantity $\|x(\vartheta, t^*, x^*, \eta^\circ) + s^\circ\|$ is by its definition not greater than the quantity $\|x(\vartheta, t^*, x^*, \eta^{**}) + s^{**}\|$, we conclude that we can find a function $\varphi^*(\delta)$ satisfying the condition $\lim \varphi^*(\delta) = 0$ as $\delta \to 0$ and such that the point $x = x(\vartheta, t^*, x^{**}, \eta^\circ) + s^\circ$ lies in the ε^*-neighborhood of the point $x = 0$, where $\varepsilon^* = \varepsilon + \varphi^*(\delta)\,\delta$, if only the region $G_M(t^*, x^{**}, \vartheta, \{\eta, [t^*, \vartheta), x^*\}_\Pi^{\circ\circ})$ intersects the closed ε_*-neighborhood of the point $x = 0$.

But this signifies that the closed convex set Z_* contains set $Y(x^*)$ and, therefore, set Z contains the closed convex hull $Y^*(x^*)$ of set $Y(x^*)$. However, by the same token we have proved that for any point $x \in Y^*(x^*)$ we can find at least one control $\eta = \eta(dt, du, dv$, or, precisely, the control $\eta = \eta^\circ(dt, du, dv \mid [t^*, \vartheta), x^*)^\circ$ which generates the motion $x(t) = x(t, t^*, x, \eta^\circ)$ arriving at the instant $t = \vartheta$ into the ε^*-neighborhood of set M, where $\varepsilon^* = \varepsilon + \varphi^*(\delta)\,\delta$. This means that regularity Condition 4.1 has been satisfied. Lemma 9.1 is proved.

The following result is obtained from Lemma 9.1 and Theorem 8.1.

Theorem 9.1. Let the function $f(t, x, u, v)$ in the right-hand side of Eq.(1.1) have continuous partial derivatives df / dx_i $(i = 1, 2, \ldots n)$ and let the following condition be fulfilled for some value $\vartheta > t_0$: if the inequality $0 < \varepsilon_0(t_*, x_*, \vartheta) < 3\beta^\circ$ is valid for a given position $\{t_*, x_*\}$ $(t_0 \leqslant t_* < \vartheta)$, then the optimal control $\eta^\circ(dt, du, dv \mid [t_*, \vartheta), x_*)^\circ$, solving Problem 3.2 for this position, is unique and the point $x_M \in M$, closest to the point $x^\circ(\vartheta)^\circ = x^\circ(\vartheta, t_*, x_*, \eta^\circ)$, is unique. Then, under the condition $\{t_0, x_0\} \in W_0$, the strategy $U^\circ \div u^\circ(t, x)$, extremal to the absorption set W_0, solves Problem 1.1. Here the inclusion

$$x[\vartheta] \in M \tag{9.3}$$

is valid for any motion $x[t] = x[t, t_0, x_0, U^\circ]$.

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